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BAYESIAN PARAMETER IDENTIFICATION IN PLASTICITY

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Abstract. To evaluate the cyclic behaviour under different loading conditions using the kinematic and isotropic hardening theory of steel a Chaboche visco-plastic material model is employed. The parameters of a constitutive model are usually identified by minimization of the distance between model response and experimental data. However, measurement errors and differences in the specimens lead to deviations in the determined parameters. In this article the Choboche model is used and a stochastic simulation technique is applied to generate artificial data which exhibit the same stochastic behaviour as experimental data. Then the model parameters are identified by applying a variaty of Bayes's theorem. Identified parameters are compared with the true parameters in the simulation and the efficiency of the identification method is discussed.

1 Introduction

In order to predict the behaviour of loaded metallic materials, constitutive models are applied, which present a mathematical frame for the description of elastic and inelastic deformation. Miller, Krempl, Korhonen, Aubertin, Chan and Bodner models can be addressed as such well-known constitutive models for isotropic materials [1, 2, 3, 4, 5]. In 1983, Chaboche [6, 7] put forward what has become known as the unified Chaboche viscoplasticity constitutive model, which has been widely accepted.

All inelastic constitutive models contain parameters which have to be identified for a given material from experiments. In the literature only few investigations can be found, dealing with identification problems using stochastic approaches. Klosowski and Mleczek have applied the least-squares method in the Marquardt-Levenberg variant to estimate the parameters of an inelastic model [8]. Gong et al. have also used some modification of the least-squares method to identify the parameters [9]. Harth and Lehn identified the model parameters of a model by employing some generated artificial data instead of experimental data using stochastic technique [10]. A similar study by Harth and Lehn has been done for other constitutive models like Lindholm and Chan [11].

In this paper, a viscoplastic model of Chaboche is studied. The model contains five material parameters which have to be determined from experimental data. It should be noted that virtual data are employed instead of real experimental data. In addition, a cyclic tension-compression test is applied in order to extract the virtual data.

Section 3 explains how to propagate the uncertainty in the model. Probabilistic model is reformulated from the deterministic model and once the forward model is provided, the model parameters are updated using a stochastic approach.

In section 4 the desired parameters are identified from the measured data. In fact, the parameters which have been considered as uncertain parameters are updated and their uncertainties are narrowed using Bayesian techniques. The results are thoroughly studied and the identified parameters as well as the corresponding model responses are analysed. Finally the prediction of the models is then compared with the measured data.

2 Model problem

The mathematical description of metals under cyclic loading beyond the yield limit that includes the viscoplastic material behaviour as well as the characterization of compulsatory isotropic-kinematic hardening is here given in terms of a modified Chaboche model introduced by [12]. As we consider classical infinitesimal material behaviour, we assume an additive strain decomposition. The material behaviour is described for the elastic part by isotropic homogeneous elasticity, and for visco-plasticity the dissipation potential is given by

$$\phi(\sigma) = \frac{k}{n+1}\left\langle \frac{\sigma_{eq} - \sigma_y}{k} \right\rangle^{n+1} = \frac{k}{n+1}\left\langle \frac{\sigma_{ex}}{k} \right\rangle^{n+1}. \tag{1}$$

with $\langle \cdot \rangle = \max(0, x)$. Here σ_{eq} is the equivalent stress which reads

$$\sigma_{eq} = \sqrt{\frac{3}{2} tr((\sigma - \chi)_D.(\sigma - \chi)_D)} \tag{2}$$

in which χ is the kinematic hardening which is defined later. $\sigma_{ex} = \sigma_{eq} - \sigma_y$ in equation 1 is the over-stress and σ_y is the yield stress. In addition, k and n in equation 1 are material parameters. The partial derivative of the dissipation potential ϕ with respect to σ leads to the equation for the inelastic strain rate

$$\dot{\epsilon}_{vp} = \frac{\partial \phi}{\partial \sigma} = \left\langle \frac{\sigma_{ex}}{k} \right\rangle^{n} \frac{\partial \sigma_{ex}}{\partial \sigma} \tag{3}$$

It should be pointed out that the over-stress σ_{ex} is the second invariant of the deviatoric stress tensor and reads the equation below.

$$\sigma_{ex} = \sigma_{eq} - \sigma_y - R = \sqrt{\frac{3}{2} tr((\sigma - \chi)_D . (\sigma - \chi)_D)} - \sigma_y - R \tag{4}$$

in which R is the isotropic hardening which is introduced in the following. The visco-plastic model allows for isotropic and kinematic hardening, which is considered in order to describe different specifications. Assuming $R(t)$ and $\chi(t)$ with $R(0) = 0$ and $\chi(0) = 0$ to describe isotropic and kinematic hardening respectively, these two are parametrised according to

$$\dot{R} = b_R(H_R - R)\dot{p} \tag{5}$$

and

$$\dot{\chi} = b_\chi(\frac{2}{3} H_\chi \frac{\partial \sigma_{eq}}{\partial \sigma} - \chi)\dot{p} \tag{6}$$

respectively. It should be mentioned that $\dot{p}$ is the visco-plastic multiplier rate given as:

$$\dot{p} = \left\langle \frac{\sigma_{ex}}{k} \right\rangle^n \tag{7}$$

which describes the rate of accumulated plastic strains. The parameter b_R indicates the speed of stabilization, whereas the value of the parameter H_R is an asymptotic value according to the evolution of the isotropic hardening. Similarly, the parameter b_χ denotes the speed of saturation and the parameter H_χ is the asymptotic value of the kinematic hardening variables. The complete model is stated in Table 1. Note that E represents the Young's modulus.

By gathering all the desired material parameters to identify into the vector $q = [\kappa\ G\ b_R\ b_\chi\ \sigma_y]$, where κ and G are bulk modulus and shear modulus, respectively, the goal is to estimate q given measurement displacement data, i.e.

$$u = Y(q) + \varepsilon \tag{8}$$

in which $Y(q)$ represents the measurement operator and ε the measurement (also possibly the model) error. Being an ill-posed problem, the estimation of q given u is not an easy task and requires regularisation. This can be achieved either in a deterministic or probabilistic setting. Here, the latter one is taken into consideration as further described in the text.

3 Bayesian identification

By acquiring additional (prior) knowledge on the parameter set next to the observation data, the probabilistic approach regularise the problem of estimating q with the help of Bayes's theorem

$$\pi_{q|u}(q|u) \propto L(q)\pi_q(q) \tag{9}$$

in which the likelihood $L(q)$ describes how likely the measurement data are given prior knowledge $\pi_q(q)$. This in turn requires the reformulation of the deterministic model into

Table 1: The constitutive model of Chaboche

Strain

$$\epsilon(t) = \epsilon_e(t) + \epsilon_{vp}(t)$$

Hooke's Law

$$\sigma(t) = E : \epsilon_e(t)$$

Flow Rule

$$\dot{\epsilon}_{vp}(t) = \langle \frac{\sigma_{eq}(t) - \sigma_y - R(t)}{k} \rangle^n \frac{\partial \sigma_{ex}}{\partial \sigma}$$

Hardening

$$\dot{R} = b_R(H_R - R)\dot{p}$$

$$\dot{\chi} = b_\chi(\frac{2}{3} H_\chi \frac{\partial \sigma_{eq}}{\partial \sigma} - \chi)\dot{p}$$

Initial Conditions

$$\epsilon_{vp}(0) = 0, \;\; R(0) = 0, \;\; \chi(0) = 0$$

Parameters

σ_y (Yield Stress)
k, n (Flow Rule)
b_R, H_R, b_χ, H_χ (Hardening)

the probabilistic one, and hence the propagation of material uncertainties through the model —the so-called forward problem— in order to obtain the likelihood [13].

The main difficulty in using equation 9 lies in computation of the likelihood. Various numerical algorithms can be applied, the most popular example of which are the Markov chain Monte Carlo methods. Being constructed on the fundamentals of the ergodic Markov theory, these methods are characterized by very slow convergence. To avoid this, the approximate method based on Kolmogorov́ definition of conditional expectation as already presented in [14] is considered here.

Let the material parameters q be modelled as random variables on a probability space $S := L_2(\Omega, \mathcal{B}, \mathbb{P})$. Here, Ω denotes the space of elementary events ω, $\mathcal{B}$ is the σ-algebra and $\mathbb{P}$ stands for the probability measure. This alternative formulation of Bayes's rule can be achieved by expressing the conditional probabilities in equation 9 in terms of conditional expectation. Following the mathematical derivation in [15, 16], this approach boils down to a quadratic minimisation problem:

$$q_a(\omega) = P_{Q_{sn}} q_f = \underset{\eta \in Q_{sn}}{\arg\min} \; \|q_f - \eta\|^2_{L_2}, \tag{10}$$

in which $P_{Q_{sn}}$ is the orthogonal projection operator of q_f onto the space of the new information $Q_{sn} := \mathcal{Q} \otimes S_n$ where the space Q_{sn} is the space of the measurement.

Constraining the space of all functions to the subspace of linear maps, the minimisation problem in equation 10 leads to a unique solution K. Note that the projection is performed over a smaller space than Q_{sn}. An implication of this is that available information is not completely used in the process of updating, introducing an approximation error. This gives an affine approximation of equation 10

$$q_a(\omega) = q_f(\omega) + K(z(\omega) - u_f(\omega)), \tag{11}$$

also known as a linear Bayesian posterior estimate. Here, q_f represents the prior random variable, q_a is the posterior approximation, u_f is the forecasted measurement and K represents the very well-known Kalman gain

$$K := C_{q_f u_f}\big(C_{u_f} + C_\varepsilon\big)^{-1} \tag{12}$$

which can be easily evaluated if the appropriate covariance matrices $C_{q_f u_f}$, C_{u_f} and C_ε are known.

An advantage of equation 11 compared to equation 9 is that the inference in equation 11 is given in terms of RVs instead of conditional densities. Namely, $q_a(\omega)$, $q_f(\omega)$, $z(\omega)$ and $u_f(\omega)$ denote the RVs used to model the posterior, prior, observation, and forecasted observation, respectively.

In this light the linear Bayesian procedure can be reduced to a simple algebraic method. Starting from the functional representation of the prior

$$\hat{q}_f = \sum_\alpha q_f^{(\alpha)} \psi_\alpha(\omega) \tag{13}$$

where ψ_α is the Hermit function. Considering the proxy in equation 13, one may discretise 11 as:

$$Q_a = Q_f + K\big(Z - U_f\big), \tag{14}$$

where $Z \in \mathbb{R}^{L \times Z}$ are the PCE coefficient of the measurement. Here, K in equation 14 is the Kalman gain evaluated in an algebraic way knowing that

$$C_{q_f,u_f} = \sum_{\alpha>0} \alpha!\; q_f^{(\alpha)} (u_f^{(\alpha)})^T. \tag{15}$$

Note that in the numerical computation $Q_f := [q_f(\omega_1), ..., q_f(\omega_Z)]$ is the PCE coefficient of the prior and $Q_a := [q_a(\omega_1), ..., q_a(\omega_Z)]$ is the PCE coefficient of the posterior with cardinality Z determined by $(L+1)$ RVs and polynomial order p. Here, the number $(L+1)$ subsumes all the RVs describing the prior and the RVs $\{\theta_i\}_{i=1}^L$ used to model the measurement error ε.

Table 2: The model parameters

κ	G	σ_y	n	k	b_R	H_R	b_χ	H_χ
1.66e9	7.69e8	1.7e8	1	1.5e8	50	0.5e8	50	0.5e8

4 Numerical results

The identification of the material constants in the Chaboche unified viscoplasticity model is a reverse process based on virtual data. In case of the Chaboche model the best way of parameters' identification is using the results of the cyclic tests, since more information can be obtained from virtual data rather than creep and relaxation tests, specifically information regarding hardening parameters. The aim of the parameter identification is to find a parameter vector q introduced in the previous section. The bulk modulus (κ), the shear modulus (G), the isotropic hardening coefficient (b_R), the kinematic hardening coefficient (b_χ) and the yield stress (σ_y) are considered as the uncertain parameters of the constitutive model.

Preliminary study is on a regular cube, modelled with one 8 node element, completely restrained on the back face, and with normal traction on the opposite (front) face. The magnitude of the normal traction and a stress in the plane of the front face is plotted in Figure 1. Purple and orange colours represent the stress value in normal and in plane directions, respectively. Considering the parameters listed in Table 2, the related σ-ϵ

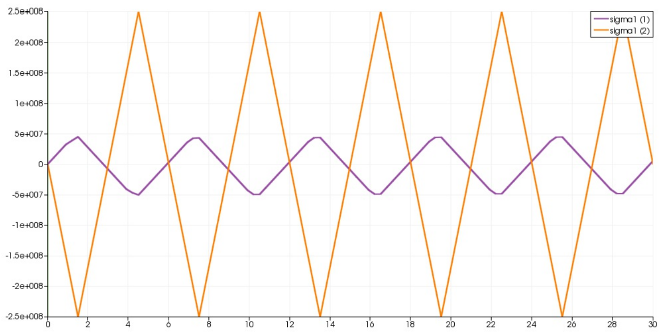

Figure 1: Decomposed applied force on node 6 according to time

hysteretic graph obtained which can be seen in Figure 2.

The displacements of a node on the front surface in normal and in plane directions are observed as the virtual data in this study. Applying stochastic identification and introducing likelihood in such a way that 10 percent of mean values are equal to the variance of the related parameter, the probability density function of prior and posterior of the identified parameters can be seen in Figure 3. From the sharpness of the posterior PDF of κ, G and σ_y, it can be concluded that enough information from virtual data is

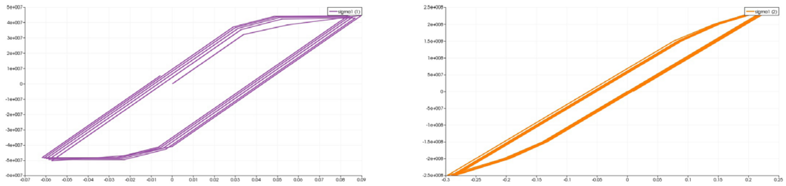

Figure 2: σ-ϵ for node on the front surface in plane and normal directions

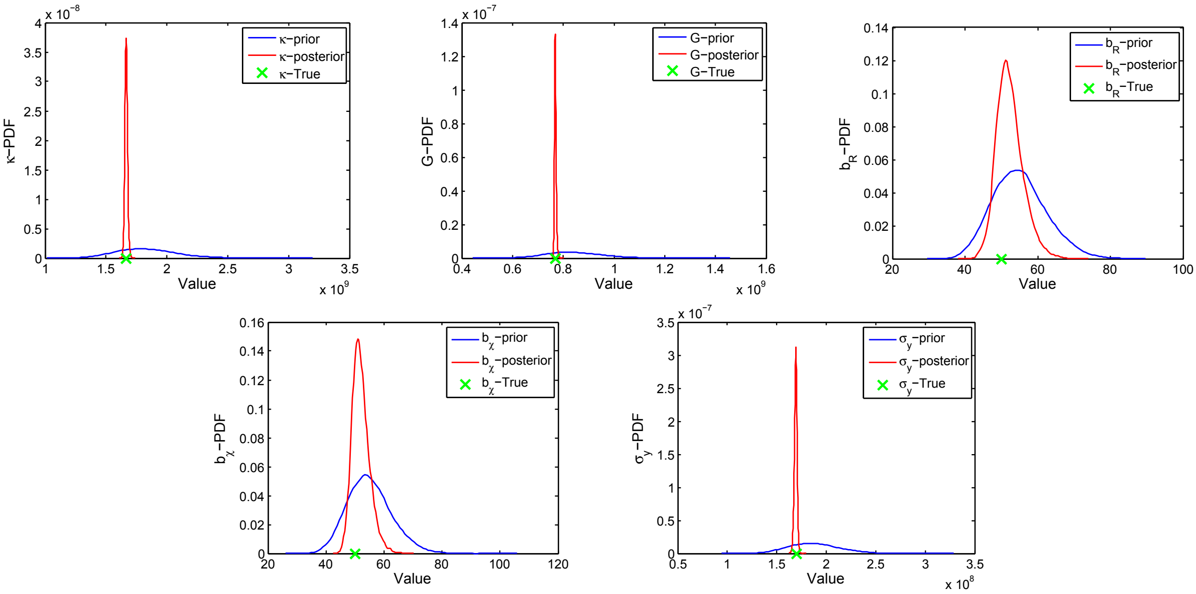

Figure 3: PDF of identified parameters

received and updating the parameters considering their uncertainty is done much easier than the hardening parameters. One reason that can be mentioned is that the process is not always in the states that hardening equations are involved like the elastic states. Therefore less information from the whole simulation can be analysed for estimating the hardening parameters and updating their parameters' uncertainties.

Summarising the results, the true values and the mean and variance of the estimated parameters are compared in Table 3.

5 Summary

Using the stochastic methods explained in section 3 to identify the model parameters of the Choboche model indicates that it is possible to identify the model parameters using Gauss-Markov Kalman filter. The parameters are well estimated and the uncertainty of the parameters is reduced while the probability density function of the parameters are updated during the process. The model is going to be developed by adding a damage model and then the efficiency of the methods used and their developments will also be studied in the near future.

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Table 3: The identified model parameters

Parameters	q_{true}	q_{est}(mean)	q_{est}(standard deviation)
κ	1.66e9	1.66e9	1.13e7
G	7.69e8	7.68e8	3.47e6
b_R	50	52.36	3.71
b_χ	50	52.04	3.01
σ_y	1.7e8	1.69e8	1.35e6

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